

FOURIER INTEGRAL OPERATORS WITH WEIGHTED SYMBOLS

ELONG OUISSAM AND SENOUSSAOUI ABDERRAHMANE

ABSTRACT. The paper contains a survey of a class of Fourier integral operators defined by symbols with tempered weight. These operators are bounded (respectively compact) in L^2 if the weight of the amplitude is bounded (respectively tends to 0).

1. INTRODUCTION

A Fourier integral operator or FIO for short has the following form

$$[I(a, \phi)f](x) = \iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} e^{i\phi(x, y, \theta)} a(x, y, \theta) f(y) dy d\theta, \quad f \in \mathcal{S}(\mathbb{R}^n) \quad (1.1)$$

where ϕ is called the *phase function* and a is the *symbol* of the FIO $I(a, \phi)$.

The study of FIO was started by considering the well known class of symbols $S_{\rho, \delta}^m$ introduced by Hörmander which consists of functions $a(x, \theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ that satisfy

$$|\partial_\theta^\alpha \partial_x^\beta a(x, \theta)| \leq C_{\alpha, \beta} (1 + |\theta|)^{m - \rho|\alpha| + \delta|\beta|},$$

with $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$. For the phase function one usually assumes that $\phi(x, \theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \setminus 0)$ is positively homogeneous of degree 1 with respect to θ and ϕ does not have critical points for $\theta \neq 0$.

Later on, other classes of symbols and phase functions were studied. In ([?]) and [?], D. Robert and B. Helffer treated the symbol class $\Gamma_\rho^\mu(\Omega)$ that consists of elements $a \in C^\infty(\Omega)$ such that for any multi-indices $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^N$, there exists $C_{\alpha, \beta, \gamma} > 0$,

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma a(x, y, \theta)| \leq C_{\alpha, \beta, \gamma} \lambda^{\mu - \rho(|\alpha| + |\beta| + |\gamma|)}(x, y, \theta),$$

where Ω is an open set of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N$, $\mu \in \mathbb{R}$ and $\rho \in [0, 1]$ and they considered phase functions satisfying certain properties. In ([?]), Messirdi and Senoussaoui treated the L^2 boundedness and L^2 compactness of FIO with symbol class just defined. These operators are continuous (respectively compact) in L^2 if the weight of the symbol is bounded (respectively tends to 0). Noted that in Hörmander's class this result is not true in general. In fact, in ([?]) the author gave an example of FIO with symbol belonging to $\bigcap_{0 < \rho < 1} S_{\rho, 1}^0$ that cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$.

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The aim of this work is to extend results obtained in ([?]), we save hypothesis on the phase function but we consider symbols with weight (m, ρ) (see below).

So in the second section we define symbol and phase functions used in this paper and we give the sense of the integral (1.1) by using the known oscillatory integral method developed by Hörmander. A special case of phase functions treated here is discussed in the preliminaries, in the third section. The last section is devoted to treat the L^2 boundedness and L^2 compactness of FIO.

2. PRELIMINARIES

Definition 1. A continuous function $m : \mathbb{R}^n \rightarrow [0, +\infty[$ is called a tempered weight on \mathbb{R}^n if

$$\exists C_0 > 0, \exists l \in \mathbb{R}; m(x) \leq C_0 m(x_1) (1 + |x_1 - x|)^l, \forall x, x_1 \in \mathbb{R}^n.$$

Functions of the form $\lambda^p(x) = (1 + |x|)^p$, $p \in \mathbb{R}$, define tempered weights.

Definition 2. Let Ω be an open set in \mathbb{R}^n , $\rho \in [0, 1]$ and m a tempered weight. A function $a \in C^\infty(\Omega)$ is called symbol with weight (m, ρ) or (m, ρ) -weighted symbol on Ω if

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0; |\partial_x^\alpha a(x)| \leq C_\alpha m(x) (1 + |x|)^{-\rho|\alpha|}, \forall x \in \Omega.$$

We note $S_\rho^m(\Omega)$ the space of symbols with weight (m, ρ) .

Proposition 2.1. Let m and l be two tempered weights.

- (i) If $a \in S_\rho^m$ then $\partial_x^\alpha \partial_\theta^\beta a \in S_\rho^{m\lambda^{-\rho|\alpha|+\beta|}};$
- (ii) If $a \in S_\rho^m$ and $b \in S_\rho^l$ then $ab \in S_\rho^{ml};$
- (iii) If $\rho \leq \delta$, $S_\delta^m \subset S_\rho^m;$
- (iv) Let $a \in S_\rho^m$. If there exists $C_0 > 0$ and $\mu \in \mathbb{R}$ such that $|a| \geq C_0 \lambda^\mu$ uniformly on Ω then $\frac{1}{a} \in S_\rho^{m\lambda^{-2\mu}}.$

Proof. For the proof we use Leibniz formula. (ii) is obtained by Leibniz formula and by induction we prove (iv). \square

Now, we consider the class of Fourier integral operators

$$[I(a, \phi)f](x) = \iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} e^{i\phi(x, y, \theta)} a(x, y, \theta) f(y) dy d\widehat{\theta}, \quad f \in \mathcal{S}(\mathbb{R}^n) \quad (2.1)$$

where $d\widehat{\theta} = (2\pi)^{-n} d\theta$, $a \in S_\rho^m$ and ϕ be a phase function which satisfies the following hypothesis

- (H1) $\phi \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^N, \mathbb{R})$ (ϕ is a real function);
- (H2) For all $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^N$, there exists $C_{\alpha, \beta, \gamma} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x, y, \theta)| \leq C_{\alpha, \beta, \gamma} \lambda^{(2-|\alpha|-|\beta|-|\gamma|)}(x, y, \theta);$$

- (H3) There exists $K_1, K_2 > 0$ such that

$$K_1 \lambda(x, y, \theta) \leq \lambda(\partial_y \phi, \partial_\theta \phi, y) \leq K_2 \lambda(x, y, \theta), \quad \forall (x, y, \theta) \in \mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^N;$$

- (H3*) There exists $K_1^*, K_2^* > 0$ such that

$$K_1^* \lambda(x, y, \theta) \leq \lambda(x, \partial_\theta \phi, \partial_x \phi) \leq K_2^* \lambda(x, y, \theta), \quad \forall (x, y, \theta) \in \mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^N.$$

To give a meaning to the right hand side of (2.1) we use the oscillatory integral method. So we consider $g \in \mathcal{S}(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^N)$ such that $g(0) = 1$. Let $a \in S_0^m$, we define

$$a_\sigma(x, y, \theta) = g\left(\frac{x}{\sigma}, \frac{y}{\sigma}, \frac{\theta}{\sigma}\right) a(x, y, \theta), \quad \sigma > 0.$$

Theorem 2.2. *Let $a \in S_0^m(\mathbb{R}^n)$ and ϕ be a phase function which satisfies (H1) – (H3). Then*

- (1) *For all $f \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{\sigma \rightarrow +\infty} [I(a_\sigma, \phi)f](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g . We define*

$$[I(a, \phi)f](x) := \lim_{\sigma \rightarrow +\infty} [I(a_\sigma, \phi)f](x);$$

- (2) *$I(a, \phi)$ defines a linear continuous operator on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ respectively.*

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$, $\text{supp } \chi \subset [-2, 2]$ such that $\chi \equiv 1$ in $[-1, 1]$. For $\varepsilon > 0$, put

$$\omega_\varepsilon(x, y, \theta) = \chi\left(\frac{|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2}{\varepsilon \lambda^2(x, y, \theta)}\right).$$

- In $\text{supp } \omega_\varepsilon$, $|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2 \leq 2\varepsilon \lambda^2(x, y, \theta)$. Using (H3) we have

$$K_1^2 \lambda^2(x, y, \theta) \leq 2\varepsilon \lambda^2(x, y, \theta) + |y|^2$$

So for ε sufficiently small, fixed at value ε_0 , we obtain

$$\lambda^2(x, y, \theta) \leq C(\varepsilon) |y|^2, \quad \forall (x, y, \theta) \in \text{supp } \omega_{\varepsilon_0}. \quad (2.2)$$

Consequently

$$\begin{aligned} |I(\omega_{\varepsilon_0} a_\sigma, \phi)f(x)| &\leq \iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} |a_\sigma(x, y, \theta)| |f(y)| dy d\theta \\ &\leq \iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} m(x, y, \theta) |f(y)| dy d\theta \end{aligned}$$

Using the definition of the tempered weight, there exists $C > 0$ and $l \in \mathbb{R}$ such that

$$|I(\omega_{\varepsilon_0} a_\sigma, \phi)f(x)| \leq C m(0, 0, 0) \iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} \lambda^l(x, y, \theta) |f(y)| dy d\theta.$$

and since $f \in \mathcal{S}(\mathbb{R}^n)$, we deduce that $I(\omega_{\varepsilon_0} a_\sigma, \phi)f$ is absolutely convergent on $\text{supp } \omega_\varepsilon$. By the Lebesgue's dominated convergence theorem we can see easily that

$$I(\omega_{\varepsilon_0} a, \phi)f = \lim_{\sigma \rightarrow +\infty} I(\omega_{\varepsilon_0} a_\sigma, \phi)f.$$

- In $\text{supp } (1 - \omega_{\varepsilon_0})$, we have

$$\text{supp } (1 - \omega_{\varepsilon_0}) \subset \Omega_0 = \{(x, y, \theta) : |\nabla_y \phi|^2 + |\nabla_\theta \phi|^2 \geq \varepsilon_0 \lambda^2(x, y, \theta)\}$$

Consider the differential operator

$$L = \frac{1}{i(|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2)} \left(\sum_{j=1}^n \frac{\partial \phi}{\partial y_j} \frac{\partial}{\partial y_j} + \sum_{k=1}^N \frac{\partial \phi}{\partial \theta_k} \frac{\partial}{\partial \theta_k} \right).$$

A basic calculus shows that $L e^{i\phi} = e^{i\phi}$.

Lemma 2.3. *For any $b \in C^\infty(\mathbb{R}_y^n \times \mathbb{R}_\theta^N)$ and any $k \in \mathbb{N}$ we have*

$$({}^tL)^k[(1 - \omega_{\varepsilon_0})b] = \sum_{|\alpha|+|\beta| \leq k} g_{\alpha,\beta}^{(k)} \partial_y^\alpha \partial_\theta^\beta ((1 - \omega_{\varepsilon_0})b)$$

where tL is the transpose operator of L and $g_{\alpha,\beta}^{(k)} \in S_0^{\lambda^{-k}}(\Omega_0)$.

Proof. The transpose operator tL has the following form

$${}^tL = \sum_{j=1}^n F_j \frac{\partial}{\partial y_j} + \sum_{j=1}^N G_j \frac{\partial}{\partial \theta_j} + H,$$

where

$$\begin{aligned} F_j &= -\frac{1}{i(|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2)} \frac{\partial \phi}{\partial y_j}, \\ G_j &= -\frac{1}{i(|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2)} \frac{\partial \phi}{\partial \theta_j}, \\ H &= -\frac{1}{i(|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2)} \left(\sum_{j=1}^n \frac{\partial^2 \phi}{\partial y_j^2} + \sum_{k=1}^N \frac{\partial^2 \phi}{\partial \theta_k^2} \right). \end{aligned}$$

In Ω_0 , $|\nabla_y \phi|^2 + |\nabla_\theta \phi|^2 \geq \varepsilon_0 \lambda^2(x, y, \theta)$ therefore using hypothesis (H2) we find

$$F_j, G_j \in S_0^{\lambda^{-1}}(\Omega_0) \text{ and } H \in S_0^{\lambda^{-2}}(\Omega_0).$$

The lemma is deduced by induction on k . □

$$I((1 - \omega_{\varepsilon_0})a_\sigma, \phi)f(x) = \iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} e^{i\phi(x, y, \theta)} ({}^tL)^k[(1 - \omega_{\varepsilon_0})a_\sigma f(y)] dy d\theta. \quad (2.3)$$

Consequently, for k large enough, the integral (2.3) converges when $\sigma \rightarrow 0$ to the absolutely convergent integral

$$\iint_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} e^{i\phi(x, y, \theta)} ({}^tL)^k[(1 - \omega_{\varepsilon_0})af(y)] dy d\theta.$$

To prove the second part, we use again the lemma (2.3). □

3. PRELIMINARIES

In the sequel we study the special phase function

$$\phi(x, y, \theta) = S(x, \theta) - y\theta. \quad (3.1)$$

where S satisfies

- (G1) $S \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\theta^n, \mathbb{R})$,
- (G2) There exists $\delta_0 > 0$ such that

$$\inf_{x, \theta \in \mathbb{R}^n} |\det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta)| \geq \delta_0.$$

- (G3) For all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exist $C_{\alpha,\beta} > 0$, such that

$$|\partial_x^\alpha \partial_\theta^\beta S(x, \theta)| \leq C_{\alpha,\beta} \lambda(x, \theta)^{(2-|\alpha|-|\beta|)}.$$

Lemma 3.1. *If S satisfies (G1), (G2) and (G3), then S satisfies (H1), (H2) and (H3). Also there exists $C > 0$ such that for all $x, x', \theta \in \mathbb{R}^n$,*

$$|x - x'| \leq C|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(x', \theta)|. \quad (3.2)$$

Proposition 3.2. *If S satisfies (G1) and (G3), then there exists a constant $\epsilon_0 > 0$ such that the phase function ϕ given in (3.1) belongs to $S_1^{\lambda^2}(\Omega_{\phi, \epsilon_0})$ where*

$$\Omega_{\phi, \epsilon_0} = \{(x, \theta, y) \in \mathbb{R}^{3n}; \quad |\partial_\theta S(x, \theta) - y|^2 < \epsilon_0 (|x|^2 + |y|^2 + |\theta|^2)\}.$$

Proof. We have to show that: There exists $\epsilon_0 > 0$, such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$, there exist $C_{\alpha, \beta, \gamma} > 0$:

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x, y, \theta)| \leq C_{\alpha, \beta, \gamma} \lambda(x, y, \theta)^{(2-|\alpha|-|\beta|-|\gamma|)}, \quad \forall (x, y, \theta) \in \Omega_{\phi, \epsilon_0}. \quad (3.3)$$

If $|\beta| = 1$, then

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x, y, \theta)| = |\partial_x^\alpha \partial_\theta^\gamma(-\theta)| = \begin{cases} 0 & \text{if } |\alpha| \neq 0 \\ |\partial_\theta^\gamma(-\theta)| & \text{if } \alpha = 0; \end{cases}$$

If $|\beta| > 1$, then $|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x, y, \theta)| = 0$.

Hence the estimate (3.3) is satisfied.

If $|\beta| = 0$, then for all $\alpha, \gamma \in \mathbb{N}^n$; $|\alpha| + |\gamma| \leq 2$, there exists $C_{\alpha, \gamma} > 0$ such that

$$|\partial_x^\alpha \partial_\theta^\gamma \phi(x, y, \theta)| = |\partial_x^\alpha \partial_\theta^\gamma S(x, \theta) - \partial_x^\alpha \partial_\theta^\gamma(y\theta)| \leq C_{\alpha, \gamma} \lambda(x, y, \theta)^{(2-|\alpha|-|\gamma|)}.$$

If $|\alpha| + |\gamma| > 2$, one has $\partial_x^\alpha \partial_\theta^\gamma \phi(x, y, \theta) = \partial_x^\alpha \partial_\theta^\gamma S(x, \theta)$. In $\Omega_{\phi, \epsilon_0}$ we have

$$|y| = |\partial_\theta S(x, \theta) - y - \partial_\theta S(x, \theta)| \leq \sqrt{\epsilon_0}(|x|^2 + |y|^2 + |\theta|^2)^{1/2} + C\lambda(x, \theta),$$

with $C > 0$. For ϵ_0 sufficiently small, we obtain a constant $C > 0$ such that

$$|y| \leq C\lambda(x, \theta), \quad \forall (x, y, \theta) \in \Omega_{\phi, \epsilon_0}. \quad (3.4)$$

This inequality leads to the equivalence

$$\lambda(x, \theta, y) \simeq \lambda(x, \theta) \quad \text{in } \Omega_{\phi, \epsilon_0} \quad (3.5)$$

thus the assumption (G3) and (3.5) give the estimate (3.3). \square

Using (3.5), we have the following result.

Proposition 3.3. *If $(x, \theta) \rightarrow a(x, \theta)$ belongs to $S_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$, then $(x, y, \theta) \rightarrow a(x, \theta)$ belongs to $S_k^{\tilde{m}}(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^n) \cap S_k^{\tilde{m}}(\Omega_{\phi, \epsilon_0})$, $k \in \{0, 1\}$.*

4. L^2 -BOUNDEDNESS AND L^2 -COMPACTNESS OF F

The main result is as follows.

Theorem 4.1. *Let F be the integral operator of distribution kernel*

$$K(x, y) = \int_{\mathbb{R}^n} e^{i(S(x, \theta) - y\theta)} a(x, \theta) d\widehat{\theta} \quad (4.1)$$

where $d\widehat{\theta} = (2\pi)^{-n} d\theta$, $a \in S_k^m(\mathbb{R}_{x, \theta}^{2n})$, $k = 0, 1$ and S satisfies (G1), (G2) and (G3). Then FF^* and F^*F are pseudodifferential operators with symbol in $S_k^{m^2}(\mathbb{R}^{2n})$, $k = 0, 1$, given by

$$\begin{aligned} \sigma(FF^*)(x, \partial_x S(x, \theta)) &\equiv |a(x, \theta)|^2 |(\det \frac{\partial^2 S}{\partial \theta \partial x})^{-1}(x, \theta)| \\ \sigma(F^*F)(\partial_\theta S(x, \theta), \theta) &\equiv |a(x, \theta)|^2 |(\det \frac{\partial^2 S}{\partial \theta \partial x})^{-1}(x, \theta)| \end{aligned}$$

we denote here $a \equiv b$ for $a, b \in S_k^{p^2}(\mathbb{R}^{2n})$ if $(a - b) \in S_k^{p^2\lambda^{-2}}(\mathbb{R}^{2n})$ and σ stands for the symbol.

Proof. If $u \in \mathcal{S}(\mathbb{R}^n)$, then $Fu(x)$ is given by

$$\begin{aligned} Fu(x) &= \int_{\mathbb{R}^n} K(x, y) u(y) dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(S(x, \theta) - y\theta)} a(x, \theta) u(y) dy d\widehat{\theta} \\ &= \int_{\mathbb{R}^n} e^{iS(x, \theta)} a(x, \theta) \left(\int_{\mathbb{R}^n} e^{-iy\theta} u(y) dy \right) d\widehat{\theta} \\ &= \int_{\mathbb{R}^n} e^{iS(x, \theta)} a(x, \theta) \mathcal{F}u(\theta) d\widehat{\theta}. \end{aligned} \quad (4.2)$$

Here F is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ (by Theorem 2.2). Let $v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned} \langle Fu, v \rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{iS(x, \theta)} a(x, \theta) \mathcal{F}u(\theta) d\widehat{\theta} \right) \overline{v(x)} dx \\ &= \int_{\mathbb{R}^n} \mathcal{F}u(\theta) \left(\int_{\mathbb{R}^n} \overline{e^{-iS(x, \theta)} a(x, \theta)} v(x) dx \right) d\widehat{\theta} \end{aligned}$$

thus

$$\langle Fu(x), v(x) \rangle_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \langle \mathcal{F}u(\theta), \mathcal{F}((F^*v))(\theta) \rangle_{L^2(\mathbb{R}^n)}$$

where

$$\mathcal{F}((F^*v))(\theta) = \int_{\mathbb{R}^n} e^{-iS(\tilde{x}, \theta)} \overline{a(\tilde{x}, \theta)} v(\tilde{x}) d\tilde{x}. \quad (4.3)$$

Hence, for all $v \in \mathcal{S}(\mathbb{R}^n)$,

$$(FF^*v)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(S(x, \theta) - S(\tilde{x}, \theta))} a(x, \theta) \overline{a(\tilde{x}, \theta)} d\tilde{x} d\widehat{\theta}. \quad (4.4)$$

The main idea to show that FF^* is a pseudodifferential operator, is to use the fact that $(S(x, \theta) - S(\tilde{x}, \theta))$ can be expressed by the scalar product $\langle x - \tilde{x}, \xi(x, \tilde{x}, \theta) \rangle$ after considering the change of variables $(x, \tilde{x}, \theta) \rightarrow (x, \tilde{x}, \xi = \xi(x, \tilde{x}, \theta))$.

The distribution kernel of FF^* is

$$K(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{i(S(x, \theta) - S(\tilde{x}, \theta))} a(x, \theta) \overline{a(\tilde{x}, \theta)} d\widehat{\theta}.$$

We obtain from (3.2) that if $|x - \tilde{x}| \geq \frac{\epsilon}{2} \lambda(x, \tilde{x}, \theta)$ (where $\epsilon > 0$ is sufficiently small) then

$$|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)| \geq \frac{\epsilon}{2C} \lambda(x, \tilde{x}, \theta). \quad (4.5)$$

Choosing $\omega \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} \omega(x) &\geq 0, \quad \forall x \in \mathbb{R} \\ \omega(x) &= 1 \quad \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \text{supp } \omega &\subset]-1, 1[\end{aligned}$$

and setting

$$\begin{aligned} b(x, \tilde{x}, \theta) &:= a(x, \theta) \overline{a}(\tilde{x}, \theta) = b_{1,\epsilon}(x, \tilde{x}, \theta) + b_{2,\epsilon}(x, \tilde{x}, \theta) \\ b_{1,\epsilon}(x, \tilde{x}, \theta) &= \omega\left(\frac{|x - \tilde{x}|}{\epsilon \lambda(x, \tilde{x}, \theta)}\right) b(x, \tilde{x}, \theta) \\ b_{2,\epsilon}(x, \tilde{x}, \theta) &= [1 - \omega\left(\frac{|x - \tilde{x}|}{\epsilon \lambda(x, \tilde{x}, \theta)}\right)] b(x, \tilde{x}, \theta). \end{aligned}$$

We have $K(x, \tilde{x}) = K_{1,\epsilon}(x, \tilde{x}) + K_{2,\epsilon}(x, \tilde{x})$, where

$$K_{j,\epsilon}(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{i(S(x,\theta) - S(\tilde{x},\theta))} b_{j,\epsilon}(x, \tilde{x}, \theta) \widehat{d\theta}, \quad j = 1, 2.$$

We will study separately the kernels $K_{1,\epsilon}$ and $K_{2,\epsilon}$.

On the support of $b_{2,\epsilon}$, inequality (4.5) is satisfied and we have

$$K_{2,\epsilon}(x, \tilde{x}) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, using the oscillatory integral method, there is a linear partial differential operator L of order 1 such that

$$L(e^{i(S(x,\theta) - S(\tilde{x},\theta))}) = e^{i(S(x,\theta) - S(\tilde{x},\theta))}$$

where

$$L = -i|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)|^{-2} \sum_{l=1}^n [(\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\tilde{x}, \theta)] \partial_{\theta_l}.$$

The transpose operator of L is

$${}^tL = \sum_{l=1}^n F_l(x, \tilde{x}, \theta) \partial_{\theta_l} + G(x, \tilde{x}, \theta)$$

where $F_l(x, \tilde{x}, \theta) \in S_0^{\lambda^{-1}}(\Omega_\epsilon)$, $G(x, \tilde{x}, \theta) \in S_0^{\lambda^{-2}}(\Omega_\epsilon)$,

$$F_l(x, \tilde{x}, \theta) = i|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)|^{-2} ((\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\tilde{x}, \theta)),$$

$$G(x, \tilde{x}, \theta) = i \sum_{l=1}^n \partial_{\theta_l} [|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)|^{-2} ((\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\tilde{x}, \theta))],$$

$$\Omega_\epsilon = \{(x, \tilde{x}, \theta) \in \mathbb{R}^{3n} : |\partial_\theta S(x, \theta) - \partial_\theta S(\tilde{x}, \theta)| > \frac{\epsilon}{2C} \lambda(x, \tilde{x}, \theta)\}.$$

On the other hand we prove by induction on q that

$$({}^tL)^q b_{2,\epsilon}(x, \tilde{x}, \theta) = \sum_{|\gamma| \leq q, \gamma \in \mathbb{N}^n} g_{\gamma,q}(x, \tilde{x}, \theta) \partial_\theta^\gamma b_{2,\epsilon}(x, \tilde{x}, \theta), \quad g_\gamma^{(q)} \in S_0^{\lambda^{-q}}(\Omega_\epsilon),$$

and so,

$$K_{2,\epsilon}(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{i(S(x,\theta) - S(\tilde{x},\theta))} ({}^tL)^q b_{2,\epsilon}(x, \tilde{x}, \theta) \widehat{d\theta}.$$

Using Leibniz's formula, (G3) and the form $({}^tL)^q$, we can choose q large enough such that for all $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n$, $\exists C_{\alpha, \alpha', \beta, \beta'} > 0$,

$$\sup_{x, \tilde{x} \in \mathbb{R}^n} |x^\alpha \tilde{x}^{\alpha'} \partial_x^\beta \partial_{\tilde{x}}^{\beta'} K_{2,\epsilon}(x, \tilde{x})| \leq C_{\alpha, \alpha', \beta, \beta'}.$$

Next, we study K_1^ϵ : this is more difficult and depends on the choice of the parameter ϵ . It follows from Taylor's formula that

$$\begin{aligned} S(x, \theta) - S(\tilde{x}, \theta) &= \langle x - \tilde{x}, \xi(x, \tilde{x}, \theta) \rangle_{\mathbb{R}^n}, \\ \xi(x, \tilde{x}, \theta) &= \int_0^1 (\partial_x S)(\tilde{x} + t(x - \tilde{x}), \theta) dt. \end{aligned}$$

We define the vectorial function

$$\tilde{\xi}_\epsilon(x, \tilde{x}, \theta) = \omega\left(\frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)}\right)\xi(x, \tilde{x}, \theta) + \left(1 - \omega\left(\frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)}\right)\right)(\partial_x S)(\tilde{x}, \theta).$$

We have

$$\tilde{\xi}_\epsilon(x, \tilde{x}, \theta) = \xi(x, \tilde{x}, \theta) \text{ on } \text{supp } b_{1, \epsilon}.$$

Moreover, for ϵ sufficiently small,

$$\lambda(x, \theta) \simeq \lambda(\tilde{x}, \theta) \simeq \lambda(x, \tilde{x}, \theta) \text{ on } \text{supp } b_{1, \epsilon}. \quad (4.6)$$

Let us consider the mapping

$$\mathbb{R}^{3n} \ni (x, \tilde{x}, \theta) \rightarrow (x, \tilde{x}, \tilde{\xi}_\epsilon(x, \tilde{x}, \theta)) \quad (4.7)$$

for which Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ \partial_x \tilde{\xi}_\epsilon & \partial_{\tilde{x}} \tilde{\xi}_\epsilon & \partial_\theta \tilde{\xi}_\epsilon \end{pmatrix}.$$

We have

$$\begin{aligned} & \frac{\partial \tilde{\xi}_{\epsilon, j}}{\partial \theta_i}(x, \tilde{x}, \theta) \\ &= \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) + \omega\left(\frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)}\right) \left(\frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right) \\ & \quad - \frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)} \frac{\partial \lambda}{\partial \theta_i}(x, \tilde{x}, \theta) \lambda^{-1}(x, \tilde{x}, \theta) \omega'\left(\frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)}\right) (\xi_j(x, \tilde{x}, \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta)). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \left| \frac{\partial \tilde{\xi}_{\epsilon, j}}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \\ & \leq \left| \omega\left(\frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)}\right) \right| \left| \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \\ & \quad + \lambda^{-1}(x, \tilde{x}, \theta) \left| \omega'\left(\frac{|x - \tilde{x}|}{2\epsilon\lambda(x, \tilde{x}, \theta)}\right) \right| \left| \xi_j(x, \tilde{x}, \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta) \right|. \end{aligned}$$

Now it follows from (G3), (4.6) and Taylor's formula that

$$\begin{aligned} \left| \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| & \leq \int_0^1 \left| \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x} + t(x - \tilde{x}), \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| dt \\ & \leq C|x - \tilde{x}| \lambda^{-1}(x, \tilde{x}, \theta), \quad C > 0 \end{aligned} \quad (4.8)$$

$$\begin{aligned} |\xi_j(x, \tilde{x}, \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta)| &\leq \int_0^1 \left| \frac{\partial S}{\partial x_j}(\tilde{x} + t(x - \tilde{x}), \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta) \right| dt \\ &\leq C|x - \tilde{x}|, \quad C > 0. \end{aligned} \quad (4.9)$$

From (4.8) and (4.9), there exists a positive constant $C > 0$ such that

$$\left| \frac{\partial \tilde{\xi}_{\epsilon, j}}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \leq C\epsilon, \quad \forall i, j \in \{1, \dots, n\}. \quad (4.10)$$

If $\epsilon < \frac{\delta_0}{2C}$, then (4.10) and (G2) yields the estimate

$$\delta_0/2 \leq -\tilde{C}\epsilon + \delta_0 \leq -\tilde{C}\epsilon + \det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \leq \det \partial_\theta \tilde{\xi}_\epsilon(x, \tilde{x}, \theta), \quad (4.11)$$

with $\tilde{C} > 0$. If ϵ is such that (4.6) and (4.11) hold, then the mapping given in (4.7) is a global diffeomorphism of \mathbb{R}^{3n} . Hence there exists a mapping

$$\theta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (x, \tilde{x}, \xi) \rightarrow \theta(x, \tilde{x}, \xi) \in \mathbb{R}^n$$

such that

$$\begin{aligned} \tilde{\xi}_\epsilon(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) &= \xi \\ \theta(x, \tilde{x}, \tilde{\xi}_\epsilon(x, \tilde{x}, \theta)) &= \theta \\ \partial^\alpha \theta(x, \tilde{x}, \xi) &= \mathcal{O}(1), \quad \forall \alpha \in \mathbb{N}^{3n} \setminus \{0\} \end{aligned} \quad (4.12)$$

If we change the variable ξ by $\theta(x, \tilde{x}, \xi)$ in $K_{1, \epsilon}(x, \tilde{x})$, we obtain

$$K_{1, \epsilon}(x, \tilde{x}) = \int_{\mathbb{R}^n} e^{i\langle x - \tilde{x}, \xi \rangle} b_{1, \epsilon}(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi}(x, \tilde{x}, \xi) \right| \widehat{d\xi}. \quad (4.13)$$

From (4.12) we have, for $k = 0, 1$, that $b_{1, \epsilon}(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi}(x, \tilde{x}, \xi) \right|$ belongs to $S_k^{m^2}(\mathbb{R}^{3n})$ if $a \in S_k^m(\mathbb{R}^{2n})$.

Applying the stationary phase theorem (c.f. [?]) to 4.13, we obtain the expression of the symbol of the pseudodifferential operator FF^* ,

$$\sigma(FF^*) = b_{1, \epsilon}(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi}(x, \tilde{x}, \xi) \right|_{\tilde{x}=x} + R(x, \xi)$$

where $R(x, \xi)$ belongs to $S_k^{m^2 \lambda^{-2}}(\mathbb{R}^{2n})$ if $a \in S_k^m(\mathbb{R}^{2n})$, $k = 0, 1$.

For $\tilde{x} = x$, we have $b_{1, \epsilon}(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) = |a(x, \theta(x, x, \xi))|^2$ where $\theta(x, x, \xi)$ is the inverse of the mapping $\theta \rightarrow \partial_x S(x, \theta) = \xi$. Thus

$$\sigma(FF^*)(x, \partial_x S(x, \theta)) \equiv |a(x, \theta)|^2 \left| \det \frac{\partial^2 S}{\partial \theta \partial x}(x, \theta) \right|^{-1}.$$

From (4.2) and (4.3), we obtain the expression of F^*F : $\forall v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} (\mathcal{F}(F^*F)\mathcal{F}^{-1})v(\theta) &= \int_{\mathbb{R}^n} e^{-iS(x, \theta)} \overline{a}(x, \theta) (F(\mathcal{F}^{-1}v))(x) dx \\ &= \int_{\mathbb{R}^n} e^{-iS(x, \theta)} \overline{a}(x, \theta) \left(\int_{\mathbb{R}^n} e^{iS(x, \tilde{\theta})} a(x, \tilde{\theta}) (\mathcal{F}(\mathcal{F}^{-1}v))(\tilde{\theta}) \widehat{d\tilde{\theta}} \right) dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(S(x, \theta) - S(x, \tilde{\theta}))} \overline{a}(x, \theta) a(x, \tilde{\theta}) v(\tilde{\theta}) \widehat{d\tilde{\theta}} dx. \end{aligned}$$

Hence the distribution kernel of the integral operator $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is

$$\tilde{K}(\theta, \tilde{\theta}) = \int_{\mathbb{R}^n} e^{-i(S(x, \theta) - S(x, \tilde{\theta}))} \widehat{a}(x, \theta) a(x, \tilde{\theta}) dx.$$

We remark that we can deduce $\tilde{K}(\theta, \tilde{\theta})$ from $K(x, \tilde{x})$ by replacing x by θ . On the other hand, all assumptions used here are symmetrical on x and θ ; therefore, $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is a nice pseudodifferential operator with symbol

$$\sigma(\mathcal{F}(F^*F)\mathcal{F}^{-1})(\theta, -\partial_\theta S(x, \theta)) \equiv |a(x, \theta)|^2 \left| \det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \right|^{-1}.$$

Thus the symbol of F^*F is given by (c.f. [?])

$$\sigma(F^*F)(\partial_\theta S(x, \theta), \theta) \equiv |a(x, \theta)|^2 \left| \det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \right|^{-1}.$$

□

Corollary 4.2. *Let F be the integral operator with the distribution kernel*

$$K(x, y) = \int_{\mathbb{R}^n} e^{i(S(x, \theta) - y\theta)} a(x, \theta) d\theta$$

where $a \in S_0^m(\mathbb{R}_{x, \theta}^{2n})$ and S satisfies (G1), (G2) and (G3). Then, we have:

- (1) For any bounded tempered weight m , F can be extended as a bounded linear mapping on $L^2(\mathbb{R}^n)$
- (2) For any m such that $\lim_{|x|+|\theta| \rightarrow \infty} m(x, \theta) = 0$, F can be extended as a compact operator on $L^2(\mathbb{R}^n)$.

Proof. It follows from Theorem 4.1 that F^*F is a pseudodifferential operator with symbol in $S_0^{m^2}(\mathbb{R}^{2n})$.

(1) Since m is bounded, we can apply the Caldéron-Vaillancourt theorem (see [?]) for F^*F and obtain the existence of a positive constant $\gamma(n)$ and an integer $k(n)$ such that

$$\|(F^*F) u\|_{L^2(\mathbb{R}^n)} \leq \gamma(n) Q_{k(n)}(\sigma(F^*F)) \|u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in \mathcal{S}(\mathbb{R}^n)$$

where

$$Q_{k(n)}(\sigma(F^*F)) = \sum_{|\alpha|+|\beta| \leq k(n)} \sup_{(x, \theta) \in \mathbb{R}^{2n}} \left| \partial_x^\alpha \partial_\theta^\beta \sigma(F^*F)(\partial_\theta S(x, \theta), \theta) \right|$$

Hence, for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\|Fu\|_{L^2(\mathbb{R}^n)} \leq \|F^*F\|_{\mathcal{L}(L^2(\mathbb{R}^n))}^{1/2} \|u\|_{L^2(\mathbb{R}^n)} \leq (\gamma(n) Q_{k(n)}(\sigma(F^*F)))^{1/2} \|u\|_{L^2(\mathbb{R}^n)}.$$

Thus F is also a bounded linear operator on $L^2(\mathbb{R}^n)$.

(2) If $\lim_{|x|+|\theta| \rightarrow \infty} m(x, \theta) = 0$, the compactness theorem (see [?]) shows that the operator F^*F can be extended as a compact operator on $L^2(\mathbb{R}^n)$. Thus, the Fourier integral operator F is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\|F^*F - \sum_{j=1}^n \langle \varphi_j, \cdot \rangle F^*F \varphi_j\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since F is bounded, for all $\psi \in L^2(\mathbb{R}^n)$,

$$\left\| F\psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle F\varphi_j \right\|^2 \leq \left\| F^* F\psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle F^* F\varphi_j \right\| \left\| \psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle \varphi_j \right\|,$$

it follows that

$$\left\| F - \sum_{j=1}^n \langle \varphi_j, \cdot \rangle F\varphi_j \right\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

□

Example 4.3. *We consider the function given by*

$$S(x, \theta) = \sum_{|\alpha|+|\beta|=2, \alpha, \beta \in \mathbb{N}^n} C_{\alpha, \beta} x^\alpha \theta^\beta, \quad \text{for } (x, \theta) \in \mathbb{R}^{2n}$$

where $C_{\alpha, \beta}$ are real constants. This function satisfies (G1), (G2) and (G3).

ELONG OUISSAM

LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN LMFAO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORAN AHMED BENBELLA, ORAN, ALGERIA

E-mail address: `elong_ouissam@yahoo.fr`

SENOUSSAOUI ABDERRAHMANE

LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN LMFAO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORAN AHMED BENBELLA, ORAN, ALGERIA

E-mail address: `senoussaoui_abdou@yahoo.fr` or `senoussaoui.abderahmane@univ-oran.dz`